XXI Appendices

A Pure altruism

In section II.B(3), pure altruism is defined solely in a verbal manner. Here, we present a formal account. Consider n agents. Agent i is endowed with private wealth W_i and considers donating D_i. One distinguishes

- the sum of all donations $D = \sum_{j=1}^{n} D_j$
- from $D_{-i} = \sum_{j=1, j \neq i}^{n} D_j$, the sum of what all the agents except for agent i donate.

Let agent i's utility (or payoff) be given by

$$[36] \quad U(C_i, D_i, D_{-i})$$

where the agent's consumption C_i equals $W_i - D_i$. According to the definition specified in the above-mentioned section, agent i is altruistic if both D_i and D_{-i} exert a positive effect on the utility of that agent:

$$[37] \quad \frac{\partial U(C_i, D_i, D_{-i})}{\partial D_i} > 0, \quad \frac{\partial U(C_i, D_i, D_{-i})}{\partial D_{-i}} > 0$$

Whenever D_i or D_{-i} increases, the overall donations increase.

A special case of altruism is called pure altruism, where the agent cares about the aggregate gift $D_{-i}+D_i$, but not about the components of this aggregate gift, i.e., whether a given amount of $D = D_{-i} + D_i$ contains a large donation by himself or a small one. This means that his utility function can be written as

$$[38] \quad U(C_i, D) = U(W_i - D_i, D_{-i} + D_i)$$

Thus, the agent exhibiting pure altruism does not distinguish between the (identical!) bundles

- $(W_i - D_i, D_{-i} + D_i)$ and
- $([W_i + \Delta] - [D_i + \Delta], [D_{-i} - \Delta] + [D_i + \Delta])$.

Assuming $\Delta > 0$ in the second bundle, the agent has greater wealth, but donates the extra wealth available to him. Thus, his consumption stays the same. His extra donation is nullified by the other agents, who donate less.

In contrast, impure altruism means that the agent derives some satisfaction from giving a large gift himself. The bundles

- $(W_i - D_i, D_i, D_{-i} + D_i)$ and
- $([W_i + \Delta] - [D_i + \Delta], D_i + \Delta, [D_{-i} - \Delta] + [D_i + \Delta])$.

are not the same. While the agent's consumption (the first entry in each bundle) and the overall donation (the third entry) are the same, the warm-glow effect (or the merit to be earned) makes it so that the agent prefers the second bundle over the first one. The question of pure or impure altruism arises only in the case of more than one donor.

For a more concrete pure-altruism utility function, consider

[39] $$U(C_i, D) = V(D_i) = (W_i - D_i)^{1-\alpha} (D_{-i} + D_i)^{\alpha}$$

with $0 \leq \alpha \leq 1$. The special case of $\alpha = 1$ amounts to extreme altruism, while $\alpha = 0$ stands for the absence of altruism. The optimal gift chosen by agent i is found by calculating the derivative of utility function V with respect to D_i, setting this derivative equal to zero, and solving for D_i:

[40] $$D_i^* = \alpha W_i - (1 - \alpha) D_{-i}$$

Understandably, the optimal gift is a positive function of an individual's wealth and a negative function of the sum of gifts given by the other agents. If private consumption is important in the utility function, i.e., if α is small, the individual tends to give a smaller portion of his private wealth as a gift and tends to reduce his gift considerably in response to an increase in others' gifts. Thus, α measures (pure) altruism in this model.

If one assumes that all n agents have the same utility function and the same amount of initial wealth, the symmetric Nash equilibrium (subsection XI.D(1)) is given by

[41] $$D_i^N = \frac{\alpha}{1 + (1 - \alpha)(n - 1)} W_i$$

The theoretically-predicted amount of an individual gift depends positively on α and negatively on n. However, the sum of all these gifts, i.e., nD_i^N, can be shown to depend positively on n if $0 < \alpha < 1$ holds.

B Matching grooms and brides in the cases of polygamy and hypergamy

This appendix refers to subsection XIV.D(2). In the model of male polygamy without, as yet, female hypergamy, the quantity of brides demanded in [9] is shown by

[42] $$\int_{\hat{m}}^{1} sm\,\mathrm{d}m = \frac{s}{2}m^2\Big|_{\hat{m}}^{1} = \frac{s}{2}\left(1 - \hat{m}^2\right)$$

In order to prove equation [10], consider a male of class $\widehat{c_V}$ with income ranging from 0 to 1. Such a male can in principle marry a woman from a class lower than $\widehat{c_V}$. The quantity of these women is $(1 - \widehat{c_V})\,w$ (multiply by 1.000 if you wish). However, some of them might already be married to higher-class men, i.e., to men with a class between 0 and $\widehat{c_V}$. Consider, now, a male from class $c_V < \widehat{c_V}$, i.e., a man who chooses wives before our male from class $\widehat{c_V}$. This type of male will marry $\frac{s}{2}\left(1 - \hat{m}^2\right)$ wives, all of whom rank lower than himself under hypergamy and where

- the portion $\frac{\widehat{c_V}-c_V}{1-c_V}$ of his wives ranks lower than $\widehat{c_V}$ and
- the portion $\frac{1-\widehat{c_V}}{1-c_V}$ of his wives ranks higher than $\widehat{c_V}$.

It is this latter portion that we need to focus on. The quantity of women from a class lower than $\widehat{c_V}$ and already married to a man from a class higher than $\widehat{c_V}$ is given by

[43] $$\int_{0}^{\widehat{c_V}} \underbrace{\frac{1 - \widehat{c_V}}{1 - c_V}}_{\substack{\text{proportion}\\ \text{of women}\\ \text{of class}\\ \text{lower than } \widehat{c_V}\\ \text{in relation}\\ \text{to women}\\ \text{of class}\\ \text{lower than } c_V}} \underbrace{\frac{s}{2}\left(1 - \hat{m}^2\right)}_{\substack{\text{quantity of wives}\\ \text{married}\\ \text{by men}\\ \text{with an income}\\ \text{above } \hat{m}}} \,\mathrm{d}c_V$$

Therefore,

[44] $$(1 - \widehat{c_V})\,w - \int_{0}^{\widehat{c_V}} \frac{1 - \widehat{c_V}}{1 - c_V}\frac{s}{2}\left(1 - \hat{m}^2\right)\mathrm{d}c_V$$

is the remaining quantity of women from whom a male of class $\widehat{c_V}$ may choose. By

[45] $$\int_{0}^{\widehat{c_V}} \frac{1}{1 - c_V}\mathrm{d}c_V = -\ln(1 - c_V)\Big|_{0}^{\widehat{c_V}} = -\ln(1 - \widehat{c_V})$$

[44] can be rewritten as

[46] $$[1 - \widehat{c_V}]\left[w + \frac{s}{2}\left(1 - \hat{m}^2\right)\ln(1 - \widehat{c_V})\right]$$

By setting [46] larger than or equal to zero, one obtains the classes of men $\widehat{c_V}$ that will be able to obtain a wife. Since $\ln(0)$ is not defined, $[46] \geq 0$ is equivalent to $\widehat{c_V} \leq 1 - e^{-\frac{2w}{s\left(1-\hat{m}^2\right)}}$.

The other, lower classes will not obtain (any fraction of) a wife. Thus, the lowest class (with the highest index) that is just able to find a wife is given by

$$[47] \quad c_V^{\min} = 1 - e^{-\frac{2w}{s\left(1-\hat{m}^2\right)}}$$

$c_V^{\min}$ has two nice properties. Firstly, $c_V^{\min} < 1$. This means that there are very low-ranked males who do not find a wife even if w is large (many potential brides), s is small (men can only support a small number of wives), and $\hat{m}$ is large (the income threshold demanded by women is large). However, taking the respective limit of these three parameters, $c_V^{\min}$ converges towards 1. Secondly, $c_V^{\min} > 0$, i.e., the highest-ranking males are sure to find a wife even if w is very small (only a few potential brides), s is large (men can support a large number of wives), and $\hat{m}$ is small (the income threshold demanded by women is small).

The two properties of being a man who (i) belongs to a class between 0 and $c_V^{\min}$ and (ii) has an income above $\hat{m}$ are assumed to be independent. Thus, the overall proportion of men finding a wife (with a strictly positive probability) equals

$$[48] \quad c_V^{\min} \cdot (1 - \hat{m}) = \left[1 - e^{-\frac{2w}{s\left(1-\hat{m}^2\right)}}\right](1 - \hat{m})$$

C Anonymous giving in a homogeneous model with productive receivers

Equation [17] in subsection XVIII.A(2)) results from DS (i.e., $rD_R = gD$) and the condition that there is no incentive to switch roles:

$$[IR] \quad \frac{g}{r}D + \ln(r) - c = U_R(D, r) \overset{!}{=} U_G(D, r) = 1 - D + \ln(r)$$

Hence, one obtains

$$[49] \quad D^{n-sw} = \frac{r}{n}(1 + c)$$

At D^{n-sw}, the payoff for each member of the society is

$$[50] \quad U_G(D^{n-sw}, g) = U_R(D^{n-sw}, g) = -c + \frac{g}{n}(1 + c) + \ln(n - g)$$

The Pareto-optimal number of givers can be found by calculating the derivative of $U_G(D^{n-sw}, g)$ with respect to the number of givers g. Setting this derivative $\frac{1+c}{n} - \frac{1}{n-g}$ equal to zero and solving for g yields

$$[51] \quad g^{opt} = n - \frac{n}{1+c} = \frac{n}{1+\frac{1}{c}} < n$$

The optimal giver-receiver ratio is constant in this model:

$$[52] \quad \frac{g^{\text{opt}}}{n} = \frac{1}{1+\frac{1}{c}} \quad \text{and} \quad \frac{r^{\text{opt}}}{n} = \frac{1}{1+c}$$

and the optimal gift turns out to be independent of c:

$$[53] \quad D^{\text{opt}} = \frac{r^{\text{opt}}}{n}(1+c) = 1$$

while the optimal gift received is not:

$$[54] \quad D_{\text{R}}^{\text{opt}} = \frac{g^{\text{opt}}}{r^{\text{opt}}} D^{\text{opt}} = \frac{g^{\text{opt}}}{r^{\text{opt}}} \frac{r^{\text{opt}}}{n}(1+c) = c$$

D A simple probabilistic model of *beneficium* reciprocity

In section XVIII.B, the optimal gift in a Seneca-inspired model is presented. Remember $D \leq 1$. Therefore, we have $\sqrt{D}W \leq W$ so that the period-1 receiver R gives at most W to period-1 giver G. The partial derivative of U^{G} with respect to D equals $-1 + \pi\tau \cdot \frac{W}{2\sqrt{D}}$. The second derivative with respect to D is obviously negative. Thus, setting this derivative equal to zero and solving for D yields the optimal gift D^{Seneca}.

E Proactive giving

This appendix shows how to solve the model of proactive giving (section XIX.H). The main information contained in Figure 21 (p. 213) is also present in the simpler Figure 24. Here, the probability of catching the potential donor's attention shows up in the payoffs.

Applying backward induction, one finds:

- After begging, giving occurs when $Ph > D_{\text{G}}$ holds.
- After not begging, giving occurs when $Ph^{+} > D_{\text{G}}$ holds.
- Let us distinguish three cases:
  - In the large-merit case of $Ph^{+} > Ph > D_{\text{G}}$, giving is always attractive to the donor. The potential receiver prefers to beg if $D_{\text{R}} - sh > \beta D_{\text{R}}$ holds, i.e., when $\beta < \frac{D_{\text{R}} - sh}{D_{\text{R}}}$.
  - In the intermediate case of $Ph^{+} > D_{\text{G}} > Ph$, giving is not attractive after begging. The potential receiver abstains from begging. Giving occurs with probability β.
  - In the low-merit case $D_{\text{G}} > Ph^{+} > Ph$, giving is never attractive. There will be neither begging nor giving.

These findings are summarised in Figure 22 (p. 213).

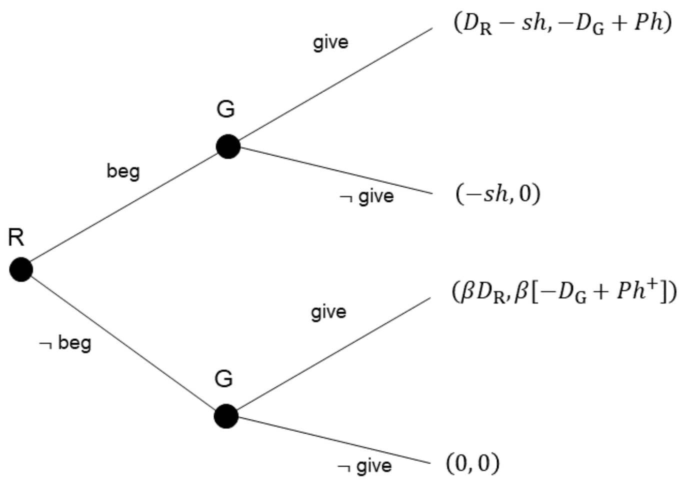

Figure 24: The proactive-giving figure simplified

F Egoistic and altruistic conflicts

In section XIX.K, some intuition behind the occurrence of an altruistic conflict has been provided. Here, a formal model is presented. It is not a game-theory model, as actions taken or strategies chosen by father and son are not modelled. I follow Stark (1993) in assuming

[55] $\mathrm{V_F}(C_\mathrm{F}) = ln(C_\mathrm{F})$

and

[56] $V_\mathrm{S}(C_\mathrm{S}) = ln(C_\mathrm{S})$

The overall consumption of corn is given by C. The two agents have to decide on how to divide $C = C_\mathrm{F} + C_\mathrm{S}$ among themselves. The father's utility can be written as

[57] $\mathrm{U_F}(C_\mathrm{F}, C_\mathrm{S}) = \beta_\mathrm{F}\mathrm{V_F}(C_\mathrm{F}) + \alpha_\mathrm{F} V_\mathrm{S}(C - C_\mathrm{F})$

We define a conflict measure

[58] $conf = \dfrac{C_\mathrm{F}^* + C_\mathrm{S}^*}{C}$

where the individually-optimal values $0 \leq C_\mathrm{F}^*, C_\mathrm{S}^* \leq 1$ are indicated by the asterix. I.e., C_F^* denotes the corn the father likes to keep for himself, while the father wants the son to enjoy $C - C_\mathrm{F}^*$ units of corn. Similarly, the son would like to have C_S^* units of corn for himself.

The conflict measure *conf* allows the following classification:

[59] $$conf = \begin{cases} < 1, & \text{altruistic conflict} \\ = 1, & \text{agreement} \\ > 1, < 2, & \text{mild egoistic conflict} \\ = 2 & \text{extreme egoistic conflict} \end{cases}$$

If the overall amount of corn that the father and the son like to consume is less than the overall endowment of corn, they are in altruistic conflict. In particular, this means $C - C_F^* > C_S^*$, i.e., the father wants the son to consume more corn than the son himself would want. Mild egoistic conflict means that one or both agents are willing to consume less than C.

From inspecting the father's utility

[60] $$U_F(C_F, C_S) = \beta_F V_F(C_F) + \alpha_F V_S(C - C_F)$$

we can derive that $\alpha_F \leq 0$ implies $C_F^* = C$ as the utility-maximising consumption level of the father. The benevolent case is more difficult. Taking the first partial derivative of U_F with respect to C_F, one obtains the first order condition

[61] $$\frac{\partial U_F}{\partial C_F} = \frac{\beta_F}{C_F} - \frac{\alpha_F}{C - C_F} = 0$$

and hence

[62] $$\left(\frac{C_F^*}{C_S}\right)_F = \frac{\beta_F}{\alpha_F}$$

The second-order condition is fulfilled by $\alpha_F \geq 0$. Similarly, the son's first-order condition is given by

[63] $$\left(\frac{C_F}{C_S^*}\right)_S = \frac{\alpha_S}{\beta_S}$$

Thus, $\alpha_F > 0$ and $\alpha_S > 0$ imply

[64] $$\left(\frac{C_F^*}{C_S}\right)_F > \left(\frac{C_F}{C_S^*}\right)_S \Longleftrightarrow \frac{\beta_F}{\alpha_F} > \frac{\alpha_S}{\beta_S} \Longleftrightarrow \beta_F \beta_S > \alpha_F \alpha_S \Longleftrightarrow conf > 1$$

The proofs of these assertions are not difficult and need not be produced here. If any of the above inequalities hold, the father wants more for himself than the son is prepared to offer.

Consider Figure 25. Depending on the level of egoism or altruism, father and son experience egoistic or altruistic conflicts. Agreement only holds for very specific combinations of parameters, i.e., when we have equalities rather than inequalities in [64]. The agreement line is in the first quadrant, where both father and son are altruistic, but not excessively altruistic. Above this line, there is altruistic conflict.

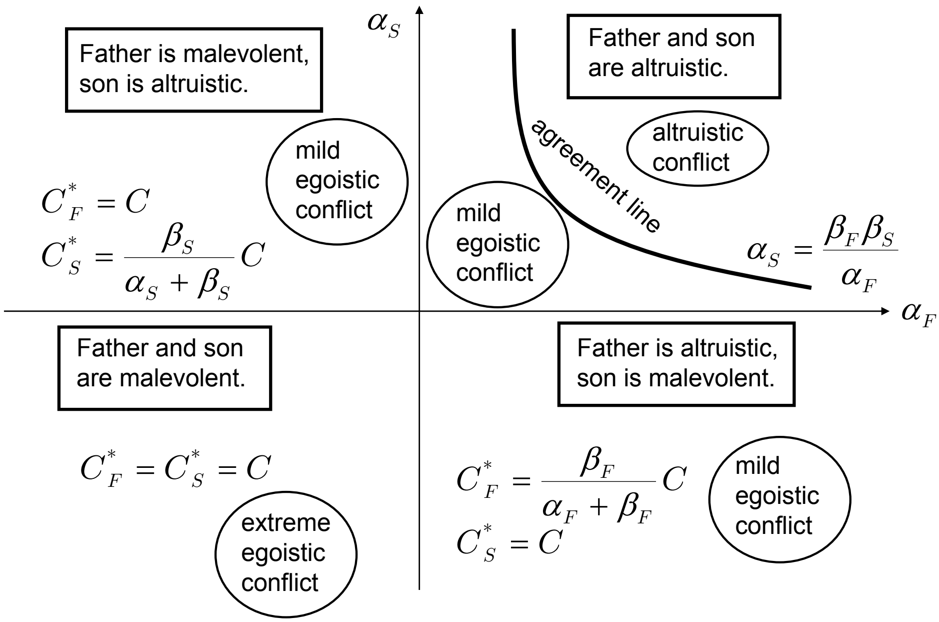

Figure 25: Types of egoistic and altruistic conflict